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Globally Stabilizing Proportional–Integral–Derivative Control Laws for Rigid-Body Attitude Tracking

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I. Introduction

ATTITUDE control of a rigid body has important applications to aircraft and spacecraft. Various control laws have been developed for pointing and slewing [1–17]. To enhance the flight envelope of a spacecraft system, globally stabilizing control laws are widely desired for attitude control.

In attitude control of a rigid-body, singular orientation and the unwinding phenomenon are two major obstacles to designing a globally stabilizing control law [1]. Every three-parameter attitude representation has an associated singular orientation, including Euler angles, Rodrigues parameters, and modified Rodrigues parameters [2]. Most of the control laws with the globally nonsingular four-parameter unit quaternion representation suffer the unwinding phenomenon. The unwinding phenomenon means that the body may start at rest arbitrarily close to the desired final attitude and yet rotate through large angles before coming to rest in the desired attitude [3].

Using rotation matrices directly could avoid the singularities and the unwinding, such as in the continuous control laws proposed in [4–8]. These control laws, however, only achieve "almost" global results, which seems to be the best possible in the sense that global stabilization using smooth feedback is not achievable.

To obtain a global result, switching control laws based on proportional and derivative (PD) feedback were proposed [9–11]. However, those control laws are not robust to arbitrarily small measurement noise. Arbitrary small measurement noise can keep system trajectories away from the target [12]. To solve this problem, hybrid PD control laws with hysteresis are proposed in [1].

Nonetheless, none of those PD control laws investigated the effects of constant disturbance torques. For those closed-loop systems with PD feedback, constant disturbance torques would result in steady-state error. To attenuate the steady-state error, very high proportional gains of PD control laws are required, which is undesirable [13].

Exploiting integral action is a main method to eliminate the steady-state error, such as with the control laws designed in [2,7,13–16]. Most of them, however, are continuous, which would not achieve global results, except the one in [15], which suffers singular orientation. The main hindrances to designing a globally stabilizing proportional–integral–derivative (PID) control law stem from the following two facts:

1) The attitude dynamics are nonlinear in nature [9,16].
2) Discontinuous control laws must be used, which makes the closed-loop system even more complicated.

In this Note, a hybrid PID control law with hysteresis is proposed, which achieves global asymptotic stability. To avoid the singular orientation, the quaternion representation is first chosen for the attitude. Then an integral term is introduced in the attitude dynamics model as a state variable. To deal with the unwinding phenomenon, the stability of the set consisting of the two equilibria must be considered, since the unwinding phenomenon results from the existence of an unstable equilibrium besides a stable equilibrium [1,10]. Then by virtue of our particular Lyapunov function and the hybrid attitude dynamic model proposed in [1], a *hysteretic hybrid PID control law* is presented. The proposed PID control law prevents the singular orientation and the unwinding phenomenon, and achieves global asymptotic tracking. Moreover, the hybrid control law is robust against measurement noises.

The remainder of this Note is organized as follows. In Sec. II, the attitude and angular velocity tracking problem is formulated. A hysteretic hybrid PID control law is designed in Sec. III. Simulation results are presented in Sec. IV. Conclusions are drawn in Sec. V.

II. Attitude Dynamics Model and Problem Formulation

A. Attitude Dynamics Model

The global nonsingular unit quaternion $\boldsymbol{q}$ is chosen to represent the attitude of the rigid body. Let $\boldsymbol{q} = [q_0 \quad \boldsymbol{q}_v^T]^T$ and $\boldsymbol{q}_v = [q_1 \quad q_2 \quad q_3]^T$. Denote the three-dimensional sphere (embedded in $\mathbb{R}^4$) by $S^3 = \{\boldsymbol{x} \in \mathbb{R}^4\colon \boldsymbol{x}^T\boldsymbol{x} = 1\}$. Then $\boldsymbol{q} \in S^3$.

In the quaternion description, the attitude kinematic and dynamic equations are the following [16]:

$$\dot{\boldsymbol{q}} = \tfrac{1}{2}B(\boldsymbol{q})\Omega \tag{1a}$$

$$\boldsymbol{J}\dot{\Omega} = -\Omega \times \boldsymbol{J}\Omega + \Gamma + \boldsymbol{d} \tag{1b}$$

where $\Omega \in \mathbb{R}^3$ is the angular velocity of the airframe in the body fixed frame, $\boldsymbol{J} \in \mathbb{R}^{3\times3}$ is the inertia matrix, $\Gamma \in \mathbb{R}^3$ is the control torque, $\boldsymbol{d} \in \mathbb{R}^3$ is an unknown constant disturbance torque. The operator $B(\boldsymbol{q})$ is given by

$$B(\boldsymbol{q}) = \begin{bmatrix} -\boldsymbol{q}_v^T \\ q_0 E^{3\times3} + \boldsymbol{q}_v^\times \end{bmatrix}$$

with identity matrix $E^{n\times n} \in \mathbb{R}^{n\times n}$. Here, $\boldsymbol{v}^\times$ is the cross-product operation and is represented in a coordinate frame by a skew-symmetric matrix:

$$\boldsymbol{v}^\times = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$$

where $\boldsymbol{v} = [v_1 \quad v_2 \quad v_3]^T$.

B. Tracking Problem Formulation

Suppose the desired attitude motion is generated by

$$\dot{\boldsymbol{q}}_d = \frac{1}{2}B(\boldsymbol{q}_d)\Omega_d$$

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where $\boldsymbol{q}_d = [q_{0d} \quad \boldsymbol{q}_{vd}^T]^T$ with $\boldsymbol{q}_{vd} = [q_{1d} \quad q_{2d} \quad q_{3d}]^T$ being the unit quaternion representing the desired attitude and $\Omega_d \in \mathbb{R}^3$ is the desired angular velocity. We assume that Ω_d and $\dot{\Omega}_d$ are bounded. This implies that the reference trajectory $s_d := \mathrm{col}(\boldsymbol{q}_d, \Omega_d, \dot{\Omega}_d)$ is bounded.

The attitude and angular velocity tracking errors are defined as follows [17]:

$$\boldsymbol{e}_v = B^T(\boldsymbol{q}_d)\boldsymbol{q} \qquad e_0 = \boldsymbol{q}_{vd}^T\boldsymbol{q}_v + q_0 q_{0d} \qquad \boldsymbol{w} = \Omega - \boldsymbol{R}\Omega_d$$

where $\boldsymbol{R} = (1 - 2\boldsymbol{e}_v^T\boldsymbol{e}_v)E^{3\times 3} + 2\boldsymbol{e}_v\boldsymbol{e}_v^T - 2e_0\boldsymbol{e}_v^\times$.

With notation $\boldsymbol{e} = [e_0 \quad \boldsymbol{e}_v^T]^T$, where $\boldsymbol{e}_v = [e_1 \quad e_2 \quad e_3]^T$, we have

$$\dot{\boldsymbol{e}} = \tfrac{1}{2}B(\boldsymbol{e})\boldsymbol{w} \tag{2a}$$

$$\boldsymbol{J}\dot{\boldsymbol{w}} = \Xi(\boldsymbol{w}, \Omega_d) + \Gamma + \boldsymbol{d} \tag{2b}$$

where $\Xi(\boldsymbol{w}, \Omega_d) = -(\boldsymbol{w} + \boldsymbol{R}\Omega_d)^\times \boldsymbol{J}(\boldsymbol{w} + \boldsymbol{R}\Omega_d) + \boldsymbol{J}(\boldsymbol{w}^\times \boldsymbol{R}\Omega_d - \boldsymbol{R}\dot{\Omega}_d)$, and $\boldsymbol{e} \in S^3$.

Therefore, the control objective is to design a state-feedback control law Γ such that, for any initial state satisfying $\boldsymbol{e}(0) \in S^3$ and $\boldsymbol{w}(0) \in \mathbb{R}^3$, the equilibrium set $\mathcal{A} = \{(\boldsymbol{e}, \boldsymbol{w}) \in S^3 \times \mathbb{R}^3: \boldsymbol{e} = \pm\boldsymbol{1}, \boldsymbol{w} = \boldsymbol{0}\}$ is asymptotically stable [1,10], where $\boldsymbol{1} = [1 \quad 0 \quad 0 \quad 0]^T$ and $\boldsymbol{0} = [0 \quad 0 \quad 0]^T$.

III. Hybrid Attitude Control Law Design

Since quaternion parameters are employed, the stability of a disconnected, two-point set in the state space is considered to avoid the unwinding phenomenon [1,10].

Let the auxiliary variable $h \in H := \{1, -1\}$ and the integral action $\boldsymbol{I}_e = \int_0^t h\boldsymbol{e}_v \, \mathrm{d}t$. The tracking control law is designed as follows:

$$\Gamma = -\Xi(\boldsymbol{w}, \Omega_d) - h(\boldsymbol{J}\boldsymbol{C}_i + c_p E^{3\times 3})\boldsymbol{e}_v - \boldsymbol{C}_d\boldsymbol{w} - \boldsymbol{C}_d\boldsymbol{C}_i\boldsymbol{I}_e \tag{3}$$

where $0 < c_p \in \mathbb{R}$, $0 < \boldsymbol{C}_i \in \mathbb{R}^{3\times 3}$, $0 < \boldsymbol{C}_d \in \mathbb{R}^{3\times 3}$ and $h^+ = -h$ when the system state belongs to $\mathcal{D}$ which is the jump set. h^+ stands for the value of h after a jump. In practice, the continuous set $\breve{\mathcal{C}}$ that indicates where continuous evolution is possible and the jump set $\mathcal{D}$ that indicates where discrete evolution is possible are designed as follows:

$$\breve{\mathcal{C}} = \{(\boldsymbol{I}_e, \boldsymbol{e}, \boldsymbol{w}, h) \in \tilde{O}: he_0 > -\eta\} \tag{4}$$

$$\mathcal{D} = \{(\boldsymbol{I}_e, \mathbf{e}, \boldsymbol{w}, h) \in \tilde{O}: he_0 \le -\eta\} \tag{5}$$

where $\tilde{O} = \mathbb{R}^3 \times S^3 \times \mathbb{R}^3 \times H$ and $\eta \in (0, 1)$.

Introducing h, $\boldsymbol{I}_e$, and Γ into the mathematical model (2), we get the hybrid closed-loop system:

$$\left.\begin{array}{c} \dot{\boldsymbol{I}}_e = h\boldsymbol{e}_v \\ \dot{\boldsymbol{e}} = \frac{1}{2}B(\boldsymbol{e})\boldsymbol{w} \\ \boldsymbol{J}\dot{\boldsymbol{w}} = -h(\boldsymbol{J}\boldsymbol{C}_i + c_p E^{3\times 3})\boldsymbol{e}_v - \boldsymbol{C}_d\boldsymbol{w} - \boldsymbol{C}_d\boldsymbol{C}_i\boldsymbol{I}_e + \boldsymbol{d} \\ \dot{h} = 0 \end{array}\right\} (\boldsymbol{I}_e, \mathbf{e}, \boldsymbol{w}, h) \in \breve{\mathcal{C}} \tag{6a}$$

$$\left.\begin{array}{c} \boldsymbol{I}_e^+ = \boldsymbol{I}_e \\ \boldsymbol{e}^+ = \boldsymbol{e} \\ \boldsymbol{w}^+ = \boldsymbol{w} \\ h^+ = -h \end{array}\right\} (\boldsymbol{I}_e, \mathbf{e}, \boldsymbol{w}, h) \in \mathcal{D} \tag{6b}$$

For compactness, let $\tilde{\boldsymbol{x}} = \mathrm{col}(\boldsymbol{x}, h)$, where $\boldsymbol{x} = \mathrm{col}(\boldsymbol{I}_e, \mathbf{e}, \boldsymbol{w})$. It should be noted here that the integral action is based on the measured attitude which may include measurement noise; thus the input of the new differential equations with the integral action is extended to $\kappa(\boldsymbol{x}, h) = \mathrm{col}(h\boldsymbol{e}_v, \boldsymbol{0}, \Gamma)$ with Γ given by Eq. (3). Then we denote Eq. (6) by

$$\mathcal{H}_{\mathrm{cl}}: \begin{cases} \dot{\tilde{\boldsymbol{x}}} = f_h(\boldsymbol{x}, \kappa(\boldsymbol{x}, h), \boldsymbol{d}) & \tilde{\boldsymbol{x}} \in \breve{\mathcal{C}} \\ \tilde{\boldsymbol{x}}^+ = g(\tilde{\boldsymbol{x}}) & \tilde{\boldsymbol{x}} \in \mathcal{D} \end{cases} \tag{7}$$

To analyze robustness of the closed-loop systems to measurement noise, generalized solutions of hybrid systems are considered [18]. The definition of generalized solutions can be found in [19], definition 3.10. Here, the generalized solutions are solutions of the hybrid system (7) with the flow set being the closure $\mathcal{C}$ of $\breve{\mathcal{C}}$:

$$\mathcal{C} = \{(\boldsymbol{I}\boldsymbol{e}, \boldsymbol{e}, \boldsymbol{w}, h) \in \tilde{O}: he_0 \ge -\eta\} \tag{8}$$

The hybrid system with $\mathcal{C}$ is considered in the following theorems.

Let $\tilde{\boldsymbol{I}}_e = \boldsymbol{I}_e - (\boldsymbol{C}_d\boldsymbol{C}_i)^{-1}\boldsymbol{d}$. The stability property of the disconnected set

$$\tilde{\mathcal{A}} = \{(\tilde{\boldsymbol{I}}_e, \boldsymbol{e}, \boldsymbol{w}, h) \in \tilde{O}: \tilde{\boldsymbol{I}}_e = \boldsymbol{0}, \boldsymbol{e} = h\boldsymbol{1}, \boldsymbol{w} = \boldsymbol{0}\}$$

is established by the following theorem.

Theorem 1: For the closed-loop system (7) with the flow set being the $\mathcal{C}$, $\tilde{\mathcal{A}}$ is globally asymptotically stable [19].

Proof: According to [19], definition 4.1, we first prove that $\tilde{\mathcal{A}}$ is prestable; then show that the maximal solutions of Eq. (7) are complete, hence $\tilde{\mathcal{A}}$ is stable; finally, verify that $\tilde{\mathcal{A}}$ is attractive and the domain of attraction is $\mathcal{B}_{\mathcal{A}} = \tilde{O}$.

1) $\tilde{\mathcal{A}}$ is prestable. Let the Lyapunov function candidate be

$$V = 2c_p(1 - he_0) + \frac{c_p}{2}\tilde{\boldsymbol{I}}_e^T\boldsymbol{C}_i\tilde{\boldsymbol{I}}_e + \frac{1}{2}(\boldsymbol{w} + \boldsymbol{C}_i\tilde{\boldsymbol{I}}_e)^T\boldsymbol{J}(\boldsymbol{w} + \boldsymbol{C}_i\tilde{\boldsymbol{I}}_e)$$

The time derivative of V along the continuous dynamics (6a) is

$$\begin{aligned} \dot{V} &= c_p h\boldsymbol{e}_v^T\boldsymbol{w} + c_p h\tilde{\boldsymbol{I}}_e^T\boldsymbol{C}_i\boldsymbol{e}_v + (\boldsymbol{w} + \boldsymbol{C}_i\tilde{\boldsymbol{I}}_e)^T(\boldsymbol{J}\dot{\boldsymbol{w}} + h\boldsymbol{J}\boldsymbol{C}_i\boldsymbol{e}_v) \\ &= (\boldsymbol{w} + \boldsymbol{C}_i\tilde{\boldsymbol{I}}_e)^T(\boldsymbol{J}\dot{\boldsymbol{w}} + h\boldsymbol{J}\boldsymbol{C}_i\boldsymbol{e}_v + c_p h\boldsymbol{e}_v) \\ &= (\boldsymbol{w} + \boldsymbol{C}_i\tilde{\boldsymbol{I}}_e)^T(-\boldsymbol{C}_d(\boldsymbol{w} + \boldsymbol{C}_i\boldsymbol{I}_e - \boldsymbol{C}_i(\boldsymbol{C}_d\boldsymbol{C}_i)^{-1}\boldsymbol{d})) \\ &= -(\boldsymbol{w} + \boldsymbol{C}_i\tilde{\boldsymbol{I}}_e)^T\boldsymbol{C}_d(\boldsymbol{w} + \boldsymbol{C}_i\tilde{\boldsymbol{I}}_e) \le 0 \end{aligned} \tag{9}$$

The change in V over jumps is

$$V(\tilde{\boldsymbol{x}}^+) - V(\tilde{\boldsymbol{x}}) = 4c_p he_0$$

In light of Eq. (5), we get

$$V(\tilde{\boldsymbol{x}}^+) - V(\tilde{\boldsymbol{x}}) \le -4c_p\eta < 0 \tag{10}$$

With [19], Theorem 4.2, in hand, we conclude that $\tilde{\mathcal{A}}$ is prestable. We note here that since $\dot{V} \le 0$ in $\mathcal{C}$ and $V(\tilde{\boldsymbol{x}}^+) - V(\tilde{\boldsymbol{x}}) < 0$ in $\mathcal{D}$, therefore, $V(t)$ is bounded by $V(0)$, i.e., $\tilde{\boldsymbol{I}}_e$ and $\boldsymbol{w}$ are all bounded.

2) The maximal generalized solutions of (7) are complete. The proof is based on [19], proposition 2.7. For each $\tilde{x} \in \mathcal{C} \cup \mathcal{D}$, if $\tilde{x} \in \mathcal{D}$ then $G(\tilde{x}) := g(\tilde{x}) \ne \emptyset$; otherwise, the condition (VC) of proposition 2.7 in [19] holds, since $\mathcal{C}\backslash\mathcal{D}$ is open and f_h is continuous and locally bounded. Then it follows by [19], proposition 2.7, that a nontrivial solution from $\tilde{x}$ exists, and since $G(\mathcal{D}) \subset \mathcal{C} \cup \mathcal{D}$, maximal solutions either are complete or have bounded hybrid time domains. Suppose that a solution $\tilde{x}(t, j)$ has a bounded hybrid time domain with T and $\bar{j}$ being the supremum of $\mathrm{dom}(\tilde{x})$ in the t and j directions, respectively. Clearly, since $G(\mathcal{D}) \subset \mathcal{C}$, if $\tilde{x}(T, \bar{j}) \in \mathcal{D}$, then the solution can be continued forward. If $\tilde{x}(T, \bar{j}) \in \mathcal{C}$ and cannot be extended forward in time, then $\tilde{x}(T, \bar{j})$ is at the boundary of $\mathcal{C}$. As mentioned before, all solutions of (7) are bounded and cannot tend to infinity, therefore, $\tilde{x}(T, \bar{j}) \in \mathcal{D} \cap \mathcal{C}$. Hence, $\tilde{x}(T, \bar{j})$ could be continued forward. Consequently, every maximal solution to $\mathcal{H}_{\mathrm{cl}}$ from $\mathcal{C} \cup \mathcal{D}$ is complete. Since $G(\mathcal{D}) \subset \mathcal{C}/\mathcal{D}$, the solutions cannot exhibit Zeno behavior (solution exhibits Zeno behavior if it is complete and $\sup_t \mathrm{dom} x < \infty$), so $t \to \infty$. Then it follows from [19], definition 4.1, that $\tilde{\mathcal{A}}$ is stable.

3) $\tilde{\mathcal{A}}$ is attractive and its domain of attraction is $\tilde{O}$. We use a hybrid version of LaSalle's invariance principle ([19], Theorem 4.14) to

prove it. Since $\{\tilde{\boldsymbol{x}} \in \mathcal{D}\colon V(\tilde{\boldsymbol{x}}^+) - V(\tilde{\boldsymbol{x}}) = 0\} = \emptyset$, it follows from [19], Theorem 4.14, that solutions converge to the largest weakly invariant set contained in

$$S = \{\boldsymbol{x} \in \mathbb{R}^3 \times S^3 \times \mathbb{R}^3\colon he_0 \geq -\eta, \dot{V} = 0\}$$

$$(\text{i.e.,}\ \ S = \{\boldsymbol{x} \in \mathbb{R}^3 \times S^3 \times \mathbb{R}^3\colon he_0 \geq -\eta, \boldsymbol{w} + \boldsymbol{C}_i \tilde{\boldsymbol{I}}_e = \boldsymbol{0}\})$$

Let $(\hat{\tilde{\boldsymbol{I}}}_e, \hat{\boldsymbol{e}}, \hat{\boldsymbol{w}}, \hat{h})$ be a solution that belongs invariantly to S. Then

$$\hat{\boldsymbol{w}} + \boldsymbol{C}_i \hat{\tilde{\boldsymbol{I}}}_e \equiv \boldsymbol{0}$$

$$\Rightarrow \dot{\hat{\boldsymbol{w}}} + \boldsymbol{C}_i \dot{\hat{\tilde{\boldsymbol{I}}}}_e \equiv \boldsymbol{0} \overset{(6a)}{\Rightarrow} \boldsymbol{J}^{-1}[-c_p h \hat{\boldsymbol{e}}_v - \boldsymbol{C}_d(\hat{\boldsymbol{w}} + \boldsymbol{C}_i \hat{\tilde{\boldsymbol{I}}}_e)] \equiv \boldsymbol{0}$$

$$\overset{(11)}{\Rightarrow} - c_p h \hat{\boldsymbol{e}}_v \equiv \boldsymbol{0} \Rightarrow \hat{\boldsymbol{e}}_v \equiv \boldsymbol{0} \overset{(6a)}{\Rightarrow} \hat{\boldsymbol{w}} \equiv \boldsymbol{0} \overset{(11)}{\Rightarrow} \hat{\tilde{\boldsymbol{I}}}_e \equiv \boldsymbol{0} \tag{11}$$

Since $h\hat{e}_0 \geq -\eta > -1$, it follows that $h\hat{e}_0 = 1$ and $\hat{\boldsymbol{e}} = h\boldsymbol{1}$. Therefore, solutions that can stay invariantly in S are $(\hat{\tilde{\boldsymbol{I}}}_e, \hat{\boldsymbol{e}}, \hat{w}, h) = (\boldsymbol{0}, h\boldsymbol{1}, \boldsymbol{0}, \pm\boldsymbol{1})$, i.e.,$\tilde{\mathcal{A}}$ is the largest invariant set. So, since solutions to the closed-loop system are complete and bounded, they converge to $\tilde{\mathcal{A}}$.

Therefore, $\tilde{\mathcal{A}}$ is globally asymptotically stable. □.

Remark 1: Though $\tilde{\boldsymbol{I}}_e$ is used in the proof, it does not appear in the control law (3). It is just an auxiliary variable for the proof. Therefore, we need not know the value of $\boldsymbol{d}$, and the control law would compensate for the constant disturbance $\boldsymbol{d}$ automatically.

Remark 2: When $h = 0$, the proof of theorem 1 fails. Since $g(\tilde{\boldsymbol{x}}) = \tilde{\boldsymbol{x}}$ with $\tilde{\boldsymbol{x}} \in \{(\tilde{\boldsymbol{I}}_e, \boldsymbol{e}, \boldsymbol{w}, h) \in \tilde{O}\colon \tilde{\boldsymbol{I}}_e = \boldsymbol{0}, e_0 = 0, \boldsymbol{w} = \boldsymbol{0}\}$, frequent switching of h would let the system stay at the switching surface forever. This problem can be avoided by defining the value of h on the switching surface $\mathcal{I} = \{(\tilde{\boldsymbol{I}}_e, \boldsymbol{e}, \boldsymbol{w}, h) \in \tilde{O}\colon e_0 = 0\}$. For example, letting $h = 1$ when $e_0 = 0$, then $h = \mathrm{sgn}(e_0)$, where

$$\mathrm{sgn}\,(e_0) = \begin{cases} 1 & e_0 \geq 0 \\ -1 & e_0 < 0 \end{cases} \tag{12}$$

The corresponding control law would be

$$\Gamma = -\Xi(\boldsymbol{w}, \Omega_d) - \mathrm{sgn}(e_0)(\boldsymbol{J}\boldsymbol{C}_i + c_p E^{3\times 3})\boldsymbol{e}_v - \boldsymbol{C}_d \boldsymbol{w} - \boldsymbol{C}_d \boldsymbol{C}_i \int_0^t \mathrm{sgn}(e_0)\boldsymbol{e}_v\,\mathrm{d}t \tag{13}$$

which is called the *signum-based control law.*

Nonetheless, with the signum-based control law arbitrary small measurement noise would let the system remain stuck at $\mathcal{I}$. At least for small measurement noise, this would not happen by using the hybrid control law with $0 < h < 1$. Actually, with the hybrid control law, the states of the closed-loop system converge to a small neighborhood of the equilibrium set $\tilde{\mathcal{A}}$ in the presence of measurement noise. This robustness property of the hybrid control law to measurement noise is stated precisely in Theorem 2 below. Before stating it, we give the definition of class $\mathcal{KL}$ [20]. A function $\beta\colon \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{KL}$ if it is continuous, the function $\beta(\cdot, t)$ at constant t is 0 at 0 and nondecreasing, while $\beta(s, \cdot)$ at constant s is nonincreasing and converges to 0 as their respective arguments increase to ∞. Denote $|\boldsymbol{x}|_{\mathcal{A}} = \inf\{\|\boldsymbol{x} - \boldsymbol{y}\|\colon \boldsymbol{y} \in \mathcal{A}\}$ as the distance from x to the set $\mathcal{A}$, where $\|\cdot\|$ is the Euclidean norm. By [20], Theorem 6.6, we get the following theorem.

Theorem 2 For given parameters $\boldsymbol{C}_i > 0$, $\boldsymbol{C}_d > 0$, $c_p > 0$ of the control law (3), there exists $\beta \in \mathcal{KL}$, and for each $\epsilon > 0$ and any compact sets $K_1 \subset \mathbb{R}^3$ and $K_2 \subset \mathbb{R}^3$, there exists $\delta > 0$, such that for each $\mu\colon \mathbb{R}_{\geq 0} \to \delta\mathbb{B}$, the solutions to the following closed-loop system:

$$\dot{\tilde{\boldsymbol{x}}} = f_h(\boldsymbol{x}, \kappa(\boldsymbol{x} + \mu, h)) \qquad (\boldsymbol{x} + \mu, h) \in \mathcal{C} \tag{14a}$$

$$\tilde{\boldsymbol{x}}^+ = g(\tilde{\boldsymbol{x}}) \qquad (\boldsymbol{x} + \mu, h) \in \mathcal{D} \tag{14b}$$

with $\tilde{\boldsymbol{x}} = \mathrm{col}(\boldsymbol{x}, h)$, and initial condition $\tilde{\boldsymbol{x}}_0 \in K_1 \times S^3 \times K_2 \times H$, satisfy

$$|\tilde{\boldsymbol{x}}(t)|_{\tilde{\mathcal{A}}} \leq \beta(|\tilde{\boldsymbol{x}}(0)|_{\tilde{\mathcal{A}}}, t) + \epsilon \quad \forall\ t \in \mathrm{dom}\ \tilde{\boldsymbol{x}}$$

IV. Simulation Results

In this section, a large-angle tracking is demonstrated through simulations of a rigid spacecraft. We use MATLAB to implement the simulations.

The rigid body considered for numerical simulations is a spacecraft [13]. Its inertia matrix is

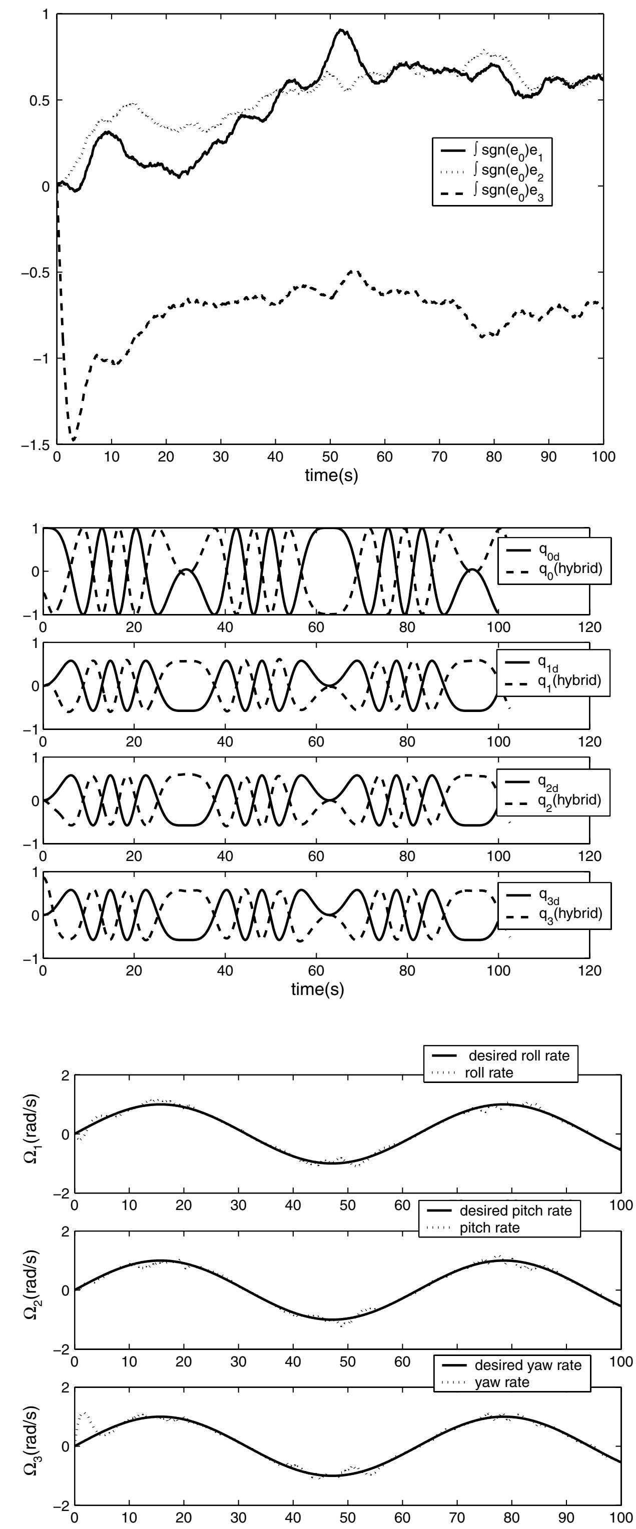

Fig. 1 Simulation results under the hybrid control law in the presence of the measurement noise and the constant disturbance torque.

$$J = \begin{bmatrix} 30 & 10 & 5 \\ 10 & 20 & 3 \\ 5 & 3 & 15 \end{bmatrix} \text{kg} \cdot \text{m}^2$$

Two simulations are conducted.

1) For the trajectory-tracking simulation, the initial parameters are chosen as $\boldsymbol{q}(0) = [-0.5 \quad 0 \quad 0 \quad 0.8660]^T$, $\Omega(0) = \boldsymbol{0}$, $\boldsymbol{q}_d(0) = [1 \quad 0 \quad 0 \quad 0]^T$, and $\boldsymbol{I}_e(0) = \boldsymbol{0}$. The desired velocity is

$$\Omega_d(t) = [\sin(0.1t) \quad \sin(0.1t) \quad \sin(0.1t)]^T$$

and $\boldsymbol{d} = [1 \quad 1 \quad -1]^T$ N · m (unknown to the control law). The control law parameters are chosen as $\boldsymbol{C}_i = 0.1E^{3\times3}$, $\boldsymbol{C}_d = 15E^{3\times3}$, $c_p = 25$, and $\eta = 0.11$ (hysteretic). The system was subjected to measurement noise so that the measured state $\tilde{\boldsymbol{q}}$ satisfies $\tilde{\boldsymbol{q}} = (\boldsymbol{q} + \mu)/\|\boldsymbol{q} + \mu\|$, where $\mu \in \mathbb{R}^4$. Denote the element in the ith row of μ by μ_i, then let μ_i, $i = 0, 1, 2, 3$, be selected randomly from a uniform distribution on $[-0.1 \quad 0.1]$ respectively. No measurement noise is used on Ω. Simulation results are shown in Fig. 1. The outputs $\mathbf{q}$ and Ω in Fig. 1 show that $\boldsymbol{q}$ tracks $-\boldsymbol{q}_d$ and that Ω tracks Ω_d with bounded error, which illustrates that the body tracks the desired attitude with bounded error. From the output of q_0, we see that q_0 tracks by the shorter path to get to the desired attitude. The integral term $\tilde{\boldsymbol{I}}_e$ shown in Fig. 1 converges to a neighborhood of

$$(\boldsymbol{C}_d\boldsymbol{C}_i)^{-1}\boldsymbol{d} = [0.667 \quad 0.667 \quad -0.667]^T$$

which illustrates that the integral term is trying to compensate the disturbance torque automatically.

In this case, outputs with the signum-based control law are almost the same as the ones shown in Fig. 1. However there exist noises which are so adversarial that they keep the trajectory near the discontinuity. We will show this in the next simulation.

2) For comparison of the signum-based control law and the hybrid control law with a certain measurement noise, the initial parameters are chosen as $\boldsymbol{q}(0) = [0 \quad 0 \quad 0 \quad 1]^T$, $\boldsymbol{q}_d(t) \equiv [0.9962 \quad 0 \quad 0.08714 \quad 0]^T$ for $t \geq 0$, and $\boldsymbol{I}_e(0) = [0.6667 \quad 0.6667 \quad -0.6667]^T$. The measurement noise is selected as $\mu_i = 0$ $(i = 1, 2, 3)$ and $\mu_0 = -2q_0 + \delta$ when $|q_0| < 0.1$, where δ is selected randomly from a uniform distribution on $[-0.03 \quad 0.03]$, μ_0 being selected randomly from a uniform distribution on $[-0.1 \quad 0.1]$ when $|q_0| \geq 0.1$. The other parameters are the same as experiment 1. Simulation results are shown in Fig. 2. From Fig. 2, we see that the signum-based control law fails to control the body to the desired attitude, whereas the hybrid law makes the rigid body asymptotically track the desired attitude with bounded error.

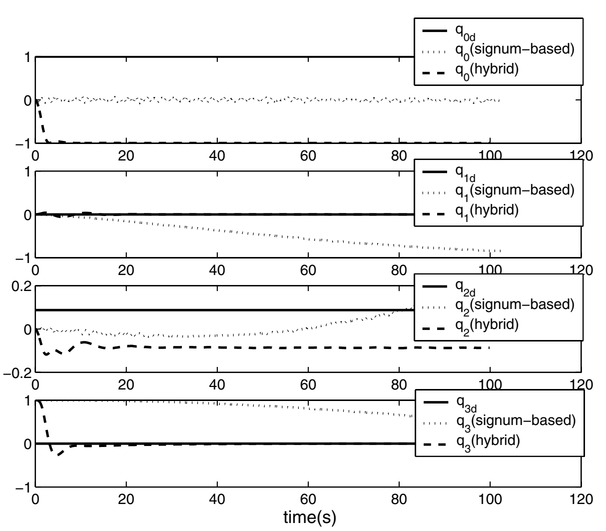

Fig. 2 Comparison of the quaternion output with the signum-based control law and the hybrid control law in the presence of the severe measurement noise and the constant disturbance torque.

V. Conclusions

This Note develops a hybrid PID attitude control law for a rigid body. It is shown that for the attitude tracking problem, it achieves globally asymptotically stable attitude trajectory tracking without the unwinding phenomenon and singular orientation, even in the presence of external constant disturbance torques. Moreover, the proposed hybrid control law is robust to measurement noise. Numerical tests were conducted, and the results demonstrate the asymptotic stability and the robustness to measurement noise.

How to design a global stable control law which is robust to inertia matrix uncertainty and disturbances is left for future work.

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